Resilient Consensus for Time-Varying Networks of Dynamic Agents

David Saldaña, Amanda Prorok, Shreyas Sundaram, Mario F. M. Campos, Vijay Kumar

Abstract— **We consider networks of dynamic agents that execute cooperative, distributed control algorithms in order to coordinate themselves and to collectively achieve goals. The agents rely on consensus algorithms that are based on local interactions with their nearest neighbors in the communication graph. However, such systems are not robust to one or more malicious agents and there are no performance guarantees when one or more agents do not cooperate. Recent results in network science deal with this problem by requiring specific graph topological properties. Nevertheless, the required network topologies imply high connectivity levels, which may be difficult to achieve in systems that exhibit time-varying communication graphs. In this paper, we propose an approach that provides resilience for networks of dynamic agents whose communication graphs are time-varying. We show that in the case where the required connectivity constraints cannot be satisfied at all times, we can resort to a consensus protocol that guarantees resilience when the union of communication graphs over a bounded period of time satisfies certain robustness properties. We propose a control policy to attain resilient behavior in the context of perimeter surveillance with a team of robots. We provide simulations that support our theoretical analyses.**

I. Introduction

The coordination of networks of dynamic agents is facilitated by group agreement strategies, i.e., consensus. Yet, the performance of such networks deteriorates if one or more agents are compromised, e.g., due to malicious attacks, or platform-level failures. For example, an attacker might take control of the communication module of certain agents in the network. As a consequence, the values communicated by the compromised agents may not correspond to the truth, and, hence, the performance of the system will be corrupted. In another example, an agent's hardware may undergo faults, subsequently influencing the values communicated by that agent. We are interested in providing resilience to such threats. In particular, we aim at solving this problem for networks of agents with communication graphs that vary as a function of time. Indeed, real-world systems with embodied agents (e.g., unmanned air vehicles (UAVs), satellites in space, vehicles in highway systems) experience constrained communication capabilities, due to limited communication ranges and physical obstruction of the communication channels. These artifacts are compounded by the motion of the agents. For these reasons, we focus on developing a resilient consensus strategy for systems where the connectivity of agents varies over time.

Our approach builds on recent work, which proposes a local consensus protocol that is resilient to a number F of non-cooperative nodes [1]. This method relies on the network topology satisfying a certain property known as $(2F+1)$-robustness. In dynamic, real-life scenarios, however, it may be difficult to achieve this property, and even impossible to maintain it throughout time — in other words, at any given moment, the network's communication graph may not satisfy required robustness properties. The premise of our work is that we can control the connectivity of individual agents in our system so that the necessary topological property is achieved *jointly*, over a collection of communication graphs in bounded time intervals. We proceed to show that such jointly connected communication topologies can be characterized by the aforementioned robustness properties. Building on this, we propose an alternate local consensus update rule that provides resilience to threats from non-cooperative agents.

A. Related Work

The topic of resilient consensus has received considerable attention, in particular in the computer science, algorithms, and engineering communities [2], [3]. A main result of this body of work states that resilience can be achieved through sufficiently high connectivity: if the connectivity of the network is $2F$ or less, then F non-cooperative nodes can prevent some of the nodes from receiving legitimate information from other nodes in the network. Conversely, when the network connectivity is $2F+1$ or higher, there are various algorithms that enable a reliable diffusion of information [4], [5]. However, these algorithms require non-local information about the (time-invariant) network topology in order to provide resilience. As a consequence, Zhang et al., and LeBlanc et al. [6], [1], [7] introduced an alternative definition of network resilience, termed r-robustness, that facilitates purely local update rules for resilient asymptotic consensus. The strength of the proposed approach is that it can deal with large networks with an arbitrary number of non-cooperative agents. Consensus within a certain 'safe' interval is assured as long as each cooperative node has no more than F non-cooperative neighbors (with $r = 2F+1$). The identities of the non-cooperative nodes are unknown to the cooperative nodes in the network and only local information is required. However, while the required network robustness properties hold in various models for large-scale networks [7], ensuring that they hold in general time-varying networks (such as those containing mobile nodes) can be challenging.

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Most similarly to our work, the recent work in [8], [9] considers asynchronous networks with information delays. That work considers the problem of finding a condition on the graph topology under which the cooperative agents reach resilient consensus with delayed information. Our work differs from the latter in that we provide an alternate (novel) update algorithm that allows nodes to compute updates at every time-step (regardless of what information has been received), and we prove that this algorithm reaches resilient consensus. Also, we provide an active control strategy that allows the network to achieve the required graph topological properties.

B. Contributions

The main contribution of this paper is a method that allows networks of dynamic agents to achieve resilient consensus when their communication graphs are time-varying. First, we demonstrate that if a team of cooperative and non-cooperative agents are linked together across a time interval by a r-robust network topology (for sufficiently large r), then the cooperative agents can achieve resilient asymptotic consensus. Second, we take advantage of the agents' motion capabilities to selectively activate communication links. We propose a control policy that moves the agents over a Jordan curve in order to ensure the robustness of the system's communication graph.

II. Preliminaries

Consider a network composed of a set of nodes $\mathcal{V} = \{1, 2, ..., n\}$. The ability to communicate with adjacent nodes defines the set of connections $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Therefore, we model the network as an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The neighbors of node i are $\mathcal{N}_i = \{j|(i,j) \in \mathcal{E}\}$. For a node subset $S \subset \mathcal{V}$, we denote its complement by $\bar{S} = \mathcal{V} \backslash S$.

A. Consensus

We consider networks of agents, where each node is an autonomous entity that can adapt to changing conditions based on incoming data streams originating from neighboring nodes. As there is no central master, the nodes need to reach an agreement with respect to the shared information in order to make unified decisions. The question of how do this is solved by *consensus algorithms* [10], [11], [12], [13]. When performing a consensus algorithm, each node i has a variable of interest x_i, e.g., that describes the locations of the nodes, or that measures local temperatures. Subsequently, the whole network may want to estimate a global variable, such as the centroid of the network, or the average temperature of the environment, respectively, based on the distributed information available to the network as a whole. This goal can be achieved by local interaction, where each node i updates its own value at time-step t based on some specified function f:

$$x_i[t+1] = f(x_i[t], \{x_j[t]|j \in \mathcal{N}_i\}). \tag{1}$$

In [11], the authors show that, if the function f represents a convex combination, then given a connected and balanced graph $\mathcal{G}$, every node $i \in \mathcal{V}$ reaches a consensus value that corresponds to a weighted average of the initial values of all nodes.

Definition 1. *An agent is said to be* cooperative *if it applies the consensus update rule and communicates its value to its neighbors at every time-step. It is called* non-cooperative *if it applies a different update rule at any time step. We denote the set of cooperative agents by $\mathcal{C} \subseteq \mathcal{V}$ and the non-cooperative agents by $\bar{\mathcal{C}} \subseteq \mathcal{V} \backslash \mathcal{C}$.*

Non-cooperative agents can be either *(i)* defective (unintentionally non-cooperative, e.g., due to a faulty sensor or actuator), or *(ii)* malicious (intentionally non-cooperative, e.g., an external attacker gains access to a node's communication module, with the goal of manipulating the system). Systems based on Eq. (1) operate well and scale well when every node is functional and cooperative. However, when a non-cooperative node stops adhering to the consensus update rule, that node can affect the behavior of all other nodes in the network. For this reason, it is desirable to devise a resilient strategy. The problem of consensus in the presence of non-cooperative nodes can be solved by deploying a *resilient consensus* algorithm.

B. Resilient Consensus

The recent work in [1] proposes a method, termed the *Weighted Mean-Subsequence-Reduced (W-MSR)* algorithm, which achieves resilient consensus by converging to a weighted average under certain topological conditions, which we detail below.

The W-MSR algorithm consists of three steps, executed at time t. First, each cooperative node i creates a sorted list, from smallest to largest, with the received values from its neighbors $\mathcal{N}_i$. Second, the list is compared to $x_i[t]$, and the F largest values that are larger than $x_i[t]$ are removed (if there are fewer than F larger values than $x_i[t]$, all of those values are removed). The same removal process is applied to the smaller values. The remaining nodes in the list are denoted by $\mathcal{R}_i[t]$. Third, node i updates its value with the following rule:

$$x_i[t+1] = w_{ii}[t]x_i[t] + \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t]x_j[t],$$

where $w_{ij}[t] > 0$, and $\sum_j w_{ij}[t] = 1$. An extended explanation of this algorithm is given in [1]. Using this algorithm, it is possible to achieve asymptotic consensus in a network with at most F non-cooperative nodes, if the communication graph $\mathcal{G}$ is $(2F+1)$-*robust*, detailed below.

Definition 2. *A set $\mathcal{S} \subseteq \mathcal{V}$ is a r-*reachable set *if there exists a node $i \in \mathcal{S}$ such that $|\mathcal{N}_i \backslash \mathcal{S}| \geq r$.*

Definition 3. *A graph $\mathcal{G}$ is r-*robust *if for any pair of nonempty, disjoint subsets of $\mathcal{V}$, at least one of them is r-*reachable.

Based on this definition of graph topology, the work in [1] shows that a network that is $(2F+1)$-robust achieves asymptotic consensus in the presence of up to F non-cooperative agents (in each local neighborhood), if cooperative agents follow the W-MSR update rule.

III. Problem Formulation

In order to ensure that deploying the W-MSR algorithm locally on each node will lead to resilient consensus in the presence of F non-cooperative agents, we must first ensure that the communication topology is $(2F+1)$-robust. For the reasons described in Section I, this may be difficult to achieve. Hence, we develop a solution that provides resilient consensus granted that the collection of joint communication graphs over a bounded time interval is $(2F+1)$-robust.

We formulate our problem as follows. Let us consider a *time-varying* graph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$, where the agents are represented by vertices, and the edges may vary through time. The neighbors of node i at time t are denoted by $\mathcal{N}_i[t] = \{j \,|(i,j) \in \mathcal{E}[t]\}$.

Problem 1. *Consider a network of dynamic agents with a time-varying communication graph, and a given threat of up to F non-cooperative agents in the network. Under the constraint that the network may not be a $(2F+1)$-robust topology at any given time-step, and under the assumption that communication links can be activated selectively (e.g., through motion control), design a strategy that allows the network to achieve asymptotic resilient consensus.*

We solve this problem in three steps. First, we demonstrate that if the agents are linked together across a time interval by a $(2F+1)$-robust network topology, then resilient consensus can be achieved. Second, we propose a sliding window approach that uses old values in order to achieve consensus based on time-varying communication graphs. Third, we propose a periodic clustering method to generate control policies that ensure that a network of mobile agents achieves robust topologies periodically. Overall, our strategy enables networks of dynamic agents to achieve resilient consensus without the need to maintain the stringent network robustness requirements at all times, but rather repeatedly, jointly over bounded time intervals.

IV. Resilient Consensus with Sliding Window

In this section, we describe our approach to solving the resilient consensus problem with jointly robust network topologies.

Definition 4. *For fixed T, $r \in \mathbb{Z}_{>0}$, the dynamic graph $\mathcal{G}[t]$ is (T,r)-robust if $\cup_{\tau=0}^{T} \mathcal{G}[t-\tau]$ satisfies the conditions of an r-robust graph $\forall t \geq T$. We denote $\mathcal{G}^T[t] := \cup_{\tau=0}^{T} \mathcal{G}[t-\tau]$.*

This implies that the dynamic graph $\mathcal{G}[t]$ is not required to be r-robust at every time step. Instead, depending on the dynamics of its edges, it may be (T,r)-robust. Based on this concept, we show that agents in our system do not need to form robust network topologies at every time-step, but rather jointly, over bounded intervals of duration T. Building on this, we extend the W-MSR algorithm [1] for time-varying networks.

A. Description of SW-MSR

We present the *Sliding Weighted Mean-Subsequence-Reduced* algorithm (SW-MSR), which introduces a sliding window approach that extends the classical W-MSR algorithm. We consider a window with duration T steps, during which each agent i stores the time-stamped values received from its neighbors. A message is a triplet $(j, \tau, x_j[\tau])$, where τ is the time-step at which agent i received the message from agent j. Algorithm 1 describes the SW-MSR update rule for each cooperative agent i at time step t. Line 2 creates a set $\mathcal{N}_i^T[t]$ that stores the neighbors during the interval $[t-T, t]$. Line 3 identifies the time-stamp of the most recent message for each neighbor j in the time interval. On line 5, the function `Remove` sorts the set $\{x_j[t-\tau_{ij}] | j \in \mathcal{N}_i^T[t]\}$ using the current agent's value $x_i[t]$ as a reference. Analogous to W-MSR, if there are less than F values strictly larger than $x_i[t]$, then agent i removes all values that are strictly larger. Otherwise, it removes precisely the largest F values in the sorted set. In the same manner, if there are less than F values strictly smaller than $x_i[t]$, then agent i removes all values that are strictly smaller. Otherwise, it removes precisely the smallest F values. It returns the non-removed elements in the set $\mathcal{R}_i[t] \subseteq \mathcal{N}_i^T[t]$. Finally, line 7 computes the weighted average based on the agent's own value and the remaining received values from the neighbors in the current time interval, where each weight is lower-bounded by $\alpha \in (0, 1/2)$ and the sum of all the weights satisfies $w_{ii}[t] + \sum_j w_{ij}[t] = 1$. Initially, while $t < T$, the agents apply the SW-MSR update rule with a temporal period $T' = t$. In this initial state, asymptotic consensus cannot be guaranteed; but the estimates are not affected by malicious outlier values since the F largest values and the F smallest values are removed in line 5.

Algorithm 1: SW-MSR$_i$(T, F, t)

1 // Find most recent values of neighboring nodes
2 $\mathcal{N}_i^T[t] := \cup_{\tau=t-T}^{t} \mathcal{N}_i[\tau]$
3 $\tau_{ij}[t] := \max(\{\tau \in [0,T] \mid j \in \mathcal{N}_i[t-\tau]\}),\ \forall j \in \mathcal{N}_i^T[t]$
4 // Remove F values strictly larger and smaller than $x_i[t]$
5 $\mathcal{R}_i[t] = \text{Remove}(x_i[t], F, \mathcal{N}_i^T[t], \tau_{ij}[t])$
6 // Compute weighted average
7 $x_i[t+1] = w_{ii}[t] x_i[t] + \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t] x_j[t - \tau_{ij}[t]]$
8 **return** $x_i[t+1]$

The main difference between this algorithm and the standard W-MSR algorithm is that we take into account all values received within the time interval T as a reference for the removal process. This feature exploits temporal information retained by each node upon the reception of messages.

B. Achieving resilient asymptotic consensus

In the following section, we demonstrate that a network with a $(T, 2F+1)$-robust topology can achieve resilient consensus in the presence of F non-cooperative nodes. This proof substantially generalizes the proofs of convergence in [1], [14] to handle the time-varying networks that we consider in this paper.

Let $M[t]$ and $m[t]$ be the maximum and the minimum values of the cooperative nodes in the time interval $[t-T, t]$, as

$$
\begin{aligned}
M[t] &:= \max_{i\in\mathcal{C},\,\tau\in[0,T]} x_i[t-\tau],\\
m[t] &:= \min_{i\in\mathcal{C},\,\tau\in[0,T]} x_i[t-\tau],\\
D[t] &:= M[t]-m[t].
\end{aligned}
$$

Lemma 1. *The functions $M[t]$ and $m[t]$ are non-increasing and non-decreasing, respectively, for $t \geq T$, if each cooperative node $i \in \mathcal{C}$ updates its current value $x_i[t+1]$ based on the SW-MSR algorithm with at most F non-cooperative nodes in the neighborhood $\mathcal{N}_i^T[t]$.*

Proof. Consider a cooperative node $i \in \mathcal{C}$, We know that after the removal process, every remaining node $(j, \tau_{ij}[t]) \in \mathcal{R}[t]$ satisfies $m[t] \leq x_j[t-\tau_{ij}[t]] \leq M[t]$. Therefore, the value for node i at the next time step is upper bounded by

$$
\begin{aligned}
x_i[t+1] &\leq w_{ii}[t]M[t] + \sum_{j\in\mathcal{R}_i[t]} w_{ij}[t]M[t]\\
&\leq M[t],
\end{aligned}
$$

and similarly, it is lower bounded by $x_i[t+1] \geq m[t]$. Since $M[t+1]$ and $m[t+1]$ are the largest and smallest values, respectively, of any cooperative node in the interval $[t-T+1, t+1]$, we see that $M[t+1] \leq M[t]$ and $m[t+1] \geq m[t]$. □

Lemma 2. *Consider a cooperative node $i \in \mathcal{C}$. For any $k \leq T$, we have $|x_i[t] - x_i[t+k]| \leq (1-\alpha^T)D[t]$*

Proof. Recall that we lower-bound the weights with $\alpha \in (0, 1/2)$. Then we have

$$
\begin{aligned}
x_i[t+1] &= w_{ii}[t]x_i[t] + \sum_{j\in\mathcal{R}_i[t]} w_{ij}[t]x_j[t-\tau_{ij}]\\
&\leq \alpha x_i[t] + (1-\alpha)M[t].
\end{aligned}
$$

For the following time step,

$$
x_i[t+2] \leq \alpha x_i[t] + (1+\alpha)(1-\alpha)M[t].
$$

After $k \leq T$ steps,

$$
\begin{aligned}
x_i[t+k] &\leq \alpha^k x_i[t] + (1+\alpha+\ldots+\alpha^{k-1})(1-\alpha)M[t]\\
&\leq \alpha^k x_i[t] + (1-\alpha^k)M[t].
\end{aligned}
$$

Then,

$$
\begin{aligned}
x_i[t] - x_i[t+k] &\geq (1-\alpha^k)(x_i[t]-M[t])\\
&\geq -(1-\alpha^k)D[t].
\end{aligned}
$$

Similarly, using the lower bound $m[t]$, we get $x_i[t] - x_i[t+k] \leq (1-\alpha^k)D[t]$ and consequently,

$$
\begin{aligned}
|x_i[t] - x_i[t+k]| &\leq (1-\alpha^k)D[t]\\
&\leq (1-\alpha^T)D[t].
\end{aligned}
$$

□

For any $\gamma \in \mathbb{R}$, let $\mathcal{X}_M(t, t', \gamma)$ be the set of cooperative nodes that had a value larger than $M[t-T]-\gamma$ at least once in the past T time steps (from t'). Similarly, let $\mathcal{X}_m(t, t', \gamma)$ be the set of cooperative nodes that had a value smaller than $m[t-T]+\gamma$ at least once in the past T time steps (from t'). Formally,

$$
\begin{aligned}
\mathcal{X}_M(t,t',\gamma) &:= \{i\in\mathcal{C} | x_i[t'-\tau] > M[t-T]-\gamma,\\
&\qquad \text{for some } 0\leq\tau\leq T\}\\
\mathcal{X}_m(t,t',\gamma) &:= \{i\in\mathcal{C} | x_i[t'-\tau] < m[t-T]+\gamma,\\
&\qquad \text{for some } 0\leq\tau\leq T\}.
\end{aligned}
$$

We will now use these sets (with appropriately defined γ) to show that the cooperative nodes reach consensus under SW-MSR dynamics and $(T, 2F+1)$-robust networks.

Theorem 1. *Given a network modeled as a graph $\mathcal{G}[t]$, resilient asymptotic consensus is achieved, even in the presence of F non-cooperative agents, if the graph is $(T, 2F+1)$-robust for every time step t, and the cooperative nodes apply the SW-MSR update rule. Furthermore, this consensus value will be in the convex hull of the initial values of the cooperative nodes.*

Proof. We will show that $D[t] \to 0$ when $t \to \infty$ by showing that for all $t \geq T$, after applying the SW-MSR update rule, the network satisfies the inequality

$$
D[t+|\mathcal{C}|(T+1)] \leq \Big(1 - \frac{\alpha^{|\mathcal{C}|(T+1)+T}}{2}\Big)D[t-T].
$$

For any $t \geq T$, consider the sets $\mathcal{X}_M(t, t, \gamma_0)$, and $\mathcal{X}_m(t, t, \gamma_0)$, with

$$
\gamma_0 := \frac{\alpha^T}{2}D[t-T].
$$

These two sets are disjoint, since a node $i \in \mathcal{X}_M(t, t, \gamma_0)$ cannot be in $\mathcal{X}_m(t, t, \gamma_0)$ (and vice versa) by Lemma 2 and the definition of γ_0.

Since $\mathcal{G}^T[t]$ is $(2F+1)$-robust, there exists a node in $\mathcal{X}_M(t, t, \gamma_0)$ or $\mathcal{X}_m(t, t, \gamma_0)$ that has $(2F+1)$ neighbors outside its set. Suppose $i \in \mathcal{X}_M(t, t, \gamma_0)$ is such a node; since there are at most F non-cooperative nodes, node i has at least $F+1$ cooperative neighbors outside the set $\mathcal{X}_M(t, t, \gamma_0)$. By the definition of this set, all of these $F+1$ cooperative neighbors had values smaller than $M[t-T]-\gamma_0$ in the interval $[t-T, t]$. If all of these $F+1$ values are smaller than $x_i[t]$, after the removal step (line 5 in Alg. 1), at least one of the remaining neighbors has a value smaller than $M[t-T]-\gamma_0$. Then the value $x_i[t+1]$ is upper bounded by

$$
\begin{aligned}
x_i[t+1] &\leq \alpha(M[t-T]-\gamma_0) + (1-\alpha)M[t]\\
&\leq M[t-T] - \alpha\gamma_0
\end{aligned}
$$

by Lemma 1. The above bound also holds if $x_i[t]$ is smaller than $M[t-T]-\gamma_0$, since node i assigns a weight of at least α to its own value (the same is in fact true for any cooperative node $i \in \mathcal{C}$ outside $\mathcal{X}_M(t, t, \gamma_0)$). By repeating this reasoning, one can show that

$$
x_i[t+k] \leq M[t-T] - \alpha^k\gamma_0
$$

for $k \in \mathbb{Z}_{>0}$. This means that node i will not be in the set $\mathcal{X}_M(t, t+T+1, \alpha^{T+1}\gamma_0)$.

Similarly, if $i \in \mathcal{X}_m(t, t, \gamma_0)$ has $2F+1$ neighbors outside its own set, then $x[t+1] \geq m[t-T] + \alpha\gamma_0$. We have the same lower bound for the next value of any other cooperative node outside $\mathcal{X}_m(t,t,\gamma_0)$. In this case, node i will not be in the set $\mathcal{X}_m(t,t+T+1,\alpha^{T+1}\gamma_0)$.

Based on the above reasoning, we have

$$\mathcal{X}_M(t,t+T+1,\alpha^{T+1}\gamma_0) \subseteq \mathcal{X}_M(t,t,\gamma_0),$$
$$\mathcal{X}_m(t,t+T+1,\alpha^{T+1}\gamma_0) \subseteq \mathcal{X}_m(t,t,\gamma_0),$$

with at least one strict inclusion, and both sets disjoint.

If both sets are nonempty, there exists a node i in one of the sets that has $2F+1$ neighbors outside that set in $\mathcal{G}^T[t+T+1]$. Following the same reasoning as above,

$$\begin{aligned}&\mathcal{X}_M(t,t+2(T+1),\alpha^{2(T+1)}\gamma_0)\\ &\qquad \subseteq \mathcal{X}_M(t,t+T+1,\alpha^{T+1}\gamma_0),\\ &\mathcal{X}_m(t,t+2(T+1),\alpha^{2(T+1)}\gamma_0)\\ &\qquad \subseteq \mathcal{X}_m(t,t+T+1,\alpha^{T+1}\gamma_0),\end{aligned}$$

with at least one strict inclusion, and both sets disjoint.

We continue in this manner by considering the sets $\mathcal{X}_M(t,t+k(T+1),\alpha^{k(T+1)}\gamma_0)$ and $\mathcal{X}_m(t,t+k(T+1),\alpha^{k(T+1)}\gamma_0)$ for $k \in \mathbb{N}$. If both sets are nonempty for some k, at least one of the sets is guaranteed to shrink in size when k is incremented. Thus, at $k=|\mathcal{C}|$, one of the sets will be empty. Suppose $\mathcal{X}_M(t,t+|\mathcal{C}|(T+1),\alpha^{|\mathcal{C}|(T+1)}\gamma_0)$ is empty. Then, by the definition of this set,

$$M[t+|\mathcal{C}|(T+1)] \leq M[t-T] - \alpha^{|\mathcal{C}|(T+1)}\gamma_0.$$

Since $m[t]$ is non-decreasing, we obtain,

$$\begin{aligned}D[t+|\mathcal{C}|(T+1)] &= M[t+|\mathcal{C}|(T+1)] - m[t+|\mathcal{C}|(T+1)]\\ &\leq D[t-T] - \alpha^{|\mathcal{C}|(T+1)}\gamma_0\\ &= D[t-T] - \frac{\alpha^{|\mathcal{C}|(T+1)+T}}{2}D[t-T]\\ &= \Big(1-\frac{\alpha^{|\mathcal{C}|(T+1)+T}}{2}\Big)D[t-T].\end{aligned}$$

The same expression will arise if the set $\mathcal{X}_m(t,t+|\mathcal{C}|(T+1),\alpha^{|\mathcal{C}|(T+1)}\gamma_0)$ is empty. Thus, as $t\to 0$, we have $D[t]\to 0$, which means that the cooperative nodes reach consensus.

The fact that the consensus value will be in the convex hull of the initial values of the cooperative nodes follows from Lemma 1. □

The derivation above shows that resilient asymptotic consensus is achieved for $(T,2F+1)$-robust graphs. In the following, we propose a control strategy that produces (T,r)-robust graphs in the context of perimeter surveillance.

V. Application to Perimeter Surveillance with Networked Robots

In applications such as the guarding of a perimeter or the surveillance of a facility, a team of multiple mobile robots is tasked to collaboratively patrol the boundary of the desired region. In order to achieve this task, the robots may have different initial estimates of the variables that allow them to successfully collaborate (e.g. minimum inter-agent distances, desired velocities, etc.), and they must rely on consensus algorithms to reach an agreement on the goal. This problem has been considered previously [15], [16], [17], even when the shape is dynamic [18]. However, all of these systems assume that all robots are cooperative.

Ring communication topologies offer a natural way to maintain network connectivity when the motion of the robots is constrained to closed curves. Importantly, the challenge posed by this task is that large perimeters and small inter-agent communication ranges may prevent the network from being connected (and thereby achieving the desired level of robustness) throughout time.

In this section, we take advantage of the agents' motion capabilities to ensure the robustness of the system's communication graph. We develop a control policy that selectively activates communication links in order to create a robust topology over time. In conjunction with our SW-MSR algorithm, we show that the cooperative robots can reach asymptotic consensus on their variables of interest.

In the Euclidean space, the location for robot i is denoted by $\mathbf{r}_i \in \mathbb{R}^2$. The communication network at time t is modeled as a time-varying graph $\mathcal{G}[t]=(\mathcal{V},\mathcal{E}[t])$, where robots are represented by vertices and the time-varying edges are computed based on a disk communication model $\mathcal{E}[t]=\{(i,j)|\ \|\mathbf{r}_j-\mathbf{r}_i\| \leq R\ \forall i,j \in \mathcal{V}\}$, where $R>0$ is the maximum communication radius. We model the perimeter of the protected region as a Jordan curve $\gamma:[0,1]\to\mathbb{R}^2$ such that $\gamma[0]=\gamma[1]$. The location of the robot i in the curve is denoted by s_i. We assume that robots are holonomic, and move in the environment based on a control input

$$u_i = \dot{s}_i.$$

In order to reconfigure the robots to achieve a (T,r)-robust communication graph, we introduce an active behavior that uses their motion capabilities along a perimeter curve $\gamma(s)$.

A. Periodic Clustering Method

Let $r \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$, and suppose there are $n=(r+1)m$ robots. We propose a periodic clustering method (PCM) that can coordinate the motion of the robots in order to create m clusters of $r+1$ elements during $r+1$ subintervals, each with duration $I>0$, where I is an upper bound on the needed time to reconfigure the robots along the curve. The goal is to aggregate the robots in such a way that all the members in each cluster can communicate among themselves (i.e., the communication graph of each cluster is a clique). We assume that the robots are indexed incrementally around the curve, as illustrated in Fig. 1(a). Then, for each subinterval, we reconfigure the members of each cluster (clique), so that a (T,r)-robust graph is achieved for the time interval $T=(r+1)I$.

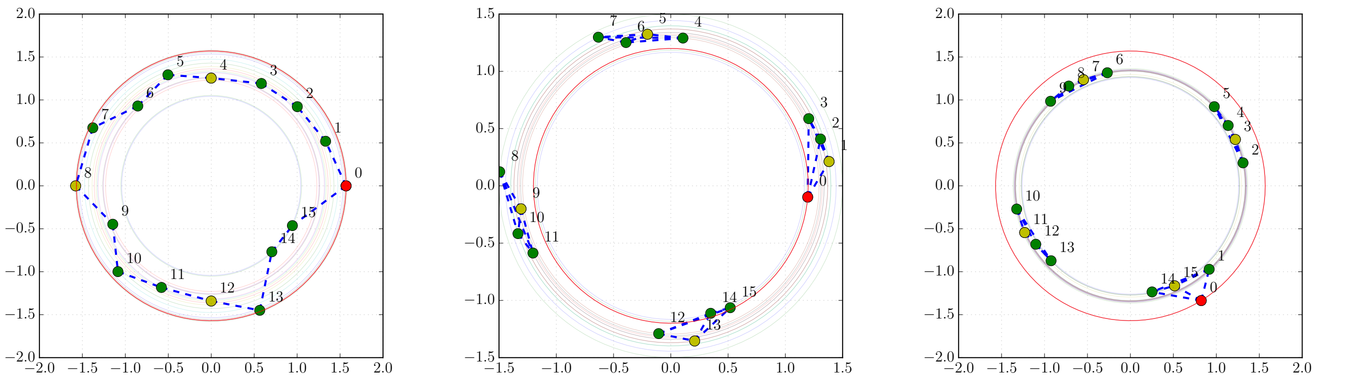
The subinterval number is computed by $\eta[t]=\lceil t/I \rceil \mod (r+1)$. For each subinterval, we define the set of m leader agents as

$$\mathcal{V}_L[t]=\{(r+1)k+\eta[t]|\ k=0,...,m-1\}.$$

For each leader $i\in\mathcal{V}_L[t]$, there are r followers

$$\mathcal{V}_{F_i}[t]=\{i+k|\ k=1,...,r\}.$$

During this subinterval, the follower nodes move towards their respective leader nodes to create a clique. This cyclic process is executed repeatedly. Specifically, for the perimeter

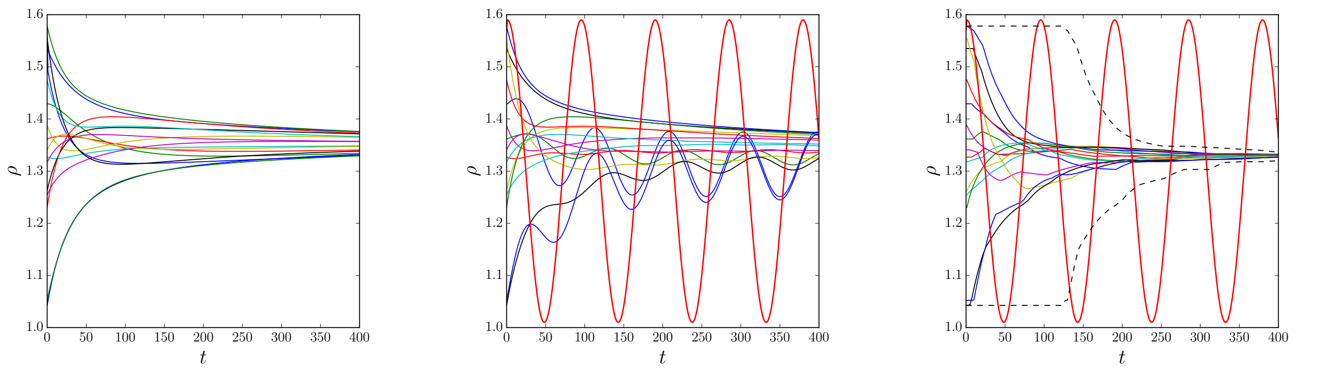


(a) At $t = 0$, the robots start in a ring configuration with only two neighbors.

(b) At $t = 30$, each leader node $i \in \{1, 6, 9, 13\}$ has its own cluster.

(c) At $t = 90$, the clusters change and their leaders are $i \in \{3, 8, 11, 15\}$.

Fig. 1. Snapshots of PCM applied to resilient perimeter surveillance. The colored disks represent the robots (yellow for the leaders, green for the followers, and red for the non-cooperative robot); the dashed lines are the communication links among the robots; the circles represent the shape function $\gamma_i[t]$ for each robot $i \in \mathcal{V}$. The periodic time period is $T = 120$.



(a) Consensus in a ring topology with two neighbors in the absence of non-cooperative agents.

(b) Consensus in a ring topology with one non-cooperative agent. The red sinusoidal line represents the value that is shared by the non-cooperative agent.

(c) Consensus using PCM. The red sinusoidal line represents the value that is shared by the non-cooperative agent. The dashed black lines represent the bounds $M[t]$ and $m[t]$.

Fig. 2. Consensus on an estimate of the circle radius ρ_i, $i \in \mathcal{V}$. With the same initial conditions, the panels show the dynamics of the consensus in three different cases.

surveillance task, where the robots circulate along the curve $\gamma(s)$, we apply the control law:

$$u_i = \Omega,$$

for each leader $i \in \mathcal{V}_L[t]$, which makes the agent patrol along the curve at a constant velocity $\Omega \in \mathbb{R}$. Each follower $j \in \mathcal{V}_{F_i}[t]$ not only patrols the curve, but also moves towards its leader using the control law:

$$u_j = \Omega + K_p\Big(s_i - s_j + (j-i)\frac{s_R}{r}\Big),$$

where $K_p > 0$ is a gain constant, and s_R is the minimum distance in the curve that satisfies the communication radius R. We assume that follower robot j can observe the location s_i of the leader robot i by a means other than communication, for example, by employing a relative positioning sensor. Overall, this behavior on the closed curve takes the robots to a robust configuration over time $t \geq (r+1)I$ [1].

Proposition 1. *If the set of mobile robots follows the Periodic Clustering Method on a closed curve $\gamma(s)$, then the communication graph of all robots is (T, r)-robust for $t \geq (r+1)I$.*

Proof. By the definition of PCM, the union of the communication graphs in the time interval $T = (k+1)I$ is:

$$\mathcal{G}^c[T] = \bigcup_{\eta=1}^{r+1} \mathcal{G}[\eta I],$$

and the periodic behavior yields $\mathcal{G}^c[t+T] = \mathcal{G}^c[t]$, $t \geq T$. In the graph $\mathcal{G}^c[t]$, each node $i \in \mathcal{V}$ has the neighbors $\mathcal{N}_i =$

[1]In our work, we assume that non-cooperative robots do not cooperate in communication, but do cooperate in motion. Despite this assumption, our threat model is sufficiently powerful, since devious motion can be easily detected with standard filtering / outlier removal methods, and non-cooperative robots flagged by their actions, whereas devious communication can be executed in such a way that it is hard to detect and ignore.

$\{i-r, ..., i-1, i+1, ..., i+r\}$, in a ring configuration where $i = n+1 = 1$.

We can show that the graph $\mathcal{G}^c[t]$ is *2r-vertex-connected*, since it is necessary to remove two subsets of r nodes in order to separate the graph into independent subgraphs. By Theorem 4 in [7], if a ring (or line) graph $\mathcal{G}$ is *k-vertex-connected* then it is at least $\lfloor k/2 \rfloor$-robust. Therefore, the graph $\mathcal{G}^c[t]$ is r-robust with period $T = (k+1)I$. □

Combining PCM and SW-MSR, given a set of ordered robots on a closed curve, we obtain a $(T, 2F+1)$-robust graph that achieves asymptotic consensus even in the presence of at most F non-cooperative robots.

B. Simulations

In our simulations we use a circular curve, where the robots try to reach consensus on the parameters of the curve. For the circle, the radius for robot i is denoted by $\rho_i > 0$, and the curve that it follows is

$$\gamma_i(s) = \rho_i \begin{bmatrix} \cos(2\pi s) \\ \sin(2\pi s) \end{bmatrix}.$$

The robots start in a ring topology, where each robot $i \in \mathcal{V}$ has two neighbors $\mathcal{N}_i[0] = \{i-1, i+1\}$ (as depicted in Fig. 1(a)). After applying PCM, we extend the neighborhood in the time interval to $\mathcal{N}_i^T[t] = \{i-r, ..., i-1, i+1, ..., i+r\}$, $t \geq T$. All cooperative agents apply the SW-MSR algorithm. In Fig. 1 we present some snapshots of the execution of PCM, where, at any given time-step, the nodes have zero or three neighbors, but over the whole time interval T, they have six neighbors. As a result, we have a ring topology with six neighbors, which is a $(T, 3)$-robust graph that can deal with $F = 1$ non-cooperative agents. The non-cooperative agent has an oscillatory behavior that tries to avoid convergence of the network.

Initially, the robots are deployed with random estimates of ρ_i (see Fig. 1(a)). If the robots start to move with constant velocity, in the absence of non-cooperative agents, in the same ring topology, they achieve asymptotic consensus as presented in Fig. 2(a). However, a single non-cooperative agent can manipulate the system, as presented in Fig. 2(b), where the non-cooperative robot shares a sinusoidal signal and avoids the convergence. Also, we note that two robots are more affected that the others: they actually correspond the neighbors of the non-cooperative agent.

Our method provides asymptotic consensus by using SW-MSR. Fig. (2(c)) demonstrates convergence. We observe the effect of the sliding window approach, as every node converges to a value that lies within the maximum and minimum of the current time interval. As time progresses, the values converge to a weighted average. We highlight that in spite of the robots' limited communication radii, PCM increases the robustness of the network to the desired level by controlling the robots' dynamics.

VI. Conclusion and Future Work

In this work, we propose a solution to providing resilience for networks of agents that have time-varying communication graphs. First, we demonstrate that if the agents are linked together across a time interval T by a $(2F+1)$-robust network topology, namely a $(T, 2F+1)$-robust topology, then resilient consensus can be achieved. Second, we propose a sliding window approach to the W-MSR algorithm, namely the SW-MSR algorithm, that enables agents to compute resilient consensus updates locally, given that their communication graphs are $(T, 2F+1)$-robust. Finally, we propose a clustering strategy that generates (T, r)-robust topologies for networked mobile agents engaged in encirclement tasks. We demonstrate the usage of our framework on an example of decentralized encirclement with robots that have limited communication radii.

In future work, we plan to extend the periodic clustering method to any type of topology, not only for ring topologies, in order to expand the number of applications in robotics and mobile communication networks.

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